数理解析研究所講究録
995 巻 1997 年 95-110

SPECTRAL GEOMETRY OF KÄHLER HYPERSURFACES IN THE COMPLEX GRASSMANN MANIFOLD

YOICHIRO MIYATA (東京都立大学 宮田洋一郎)

§1. Introduction.

Let M be a compact C^{∞}-Riemannian manifold, $C^{\infty}(M)$ the space of all smooth functions on M, and Δ the Laplacian on M. Then Δ is a self-adjoint elliptic differential operator acting on $C^{\infty}(M)$, which has an infinite discrete sequence of eigenvalues: $Spec(M) = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \uparrow \infty\}$. Let $V_k = V_k(M)$ be the eigenspace of Δ corresponding to the k-th eigenvalue λ_k. Then V_k is finite-dimensional. We define an inner product $(\ ,\ )_{L^2}$ on $C^{\infty}(M)$ by $(f, g)_{L^2} = \int_M fg\, dv_M$. where dv_M denotes the volume element on M. Then $\sum_{t=0}^{\infty} V_t$ is dense in $C^{\infty}(M)$ and the decomposition is orthogonal with respect to the inner product $(\ ,\ )_{L^2}$. Thus we have $C^{\infty}(M) = \sum_{t=0}^{\infty} V_t(M)$ (in L^2-sense). Since M is compact, V_0 is the space of all constant functions which is 1-dimensional.

In this point of view, it is one of the simplest and the most interesting problems to estimate the first eigenvalue. In [10], A. Ros gave the following sharp upper bound for the first eigenvalue of Kähler submanifold of a complex projective space.

Theorem 1.1. *Suppose that M is a complex m-dimensional compact Kähler submanifold of the complex projective space $\mathbb{C}P^n$ of constant holomorphic sectional curvature c. Then the first eigenvalue λ_1 satisfies the following inequality:*

$$\lambda_1 \leqq c(m+1)$$

The equality holds if and only if M is congruent to the totally geodesic Kähler submanifold $\mathbb{C}P^m$ of $\mathbb{C}P^n$.

If M is not totally geodesic, J-P. Bourguignon, P. Li and S. T. Yau in [1] gave the following more sharp estimate. (See also [7].)

Theorem 1.2. *Suppose that M is a complex m-dimensional compact Kähler submanifold of $\mathbb{C}P^n$, which is fully immersed and not totally geodesic. Then the first eigenvalue λ_1 satisfies the following inequality:*

$$\lambda_1 \leqq cm\frac{n+1}{n}$$

It is unknown when the equality holds in this inequality.

Our purpose is to give the upper bound for the first eigenvalue of Kähler hypersurfaces of a complex Grassmann manifold.

Let denote by $G_r(\mathbb{C}^n)$ the complex Grassmann manifold of r-planes in $\mathbb{C}^n$, equiped with the Kähler metric of maximal holomorphic sectional curvature c. We obtain the following result which is a natural generalization of Theorem 1.1.

Theorem A. *Suppose that M is a compact connected Kähler hypersurface of $G_r(\mathbb{C}^n)$. Then the first eigenvalue λ_1 satisfies the following inequality:*

$$\lambda_1 \leqq c\left(n - \frac{n-2}{r(n-r)-1}\right)$$

The equality holds if and only if $r = 1, n$, and M is congruent to the totally geodesic complex hypersurface $\mathbb{C}P^{n-2}$ of the complex projective space $\mathbb{C}P^{n-1}$.

The 2-plane Grassmann manifold $G_2(\mathbb{C}^n)$ admits the quaternionic Kähler structure $\mathfrak{J}$. For the normal bundle $T^\perp M$ of a Kähler hypersurface M of $G_2(\mathbb{C}^n)$, $\mathfrak{J}T^\perp M$ is a vector bundle of real rank 6 over M which is a subbundle of the tangent bundle of $G_2(\mathbb{C}^n)$. We consider a Kähler hypersurface M of $G_2(\mathbb{C}^n)$ satisfying the property that $\mathfrak{J}T^\perp M$ is a subbundle of the tangent bundle TM of M. In the section 4, we will introduce examples satisfying this property.

For a Kähler hypersurface of $G_2(\mathbb{C}^n)$ satisfying this property, we obtain the following upper bound of the first eigenvalue.

Theorem B. *Suppose that M is a compact connected Kähler hypersurface of $G_2(\mathbb{C}^n)$, $n \geqq 4$. If M satisfies the condition $\mathfrak{J}\, T^\perp M \subset TM$, then the following inequality holds:*

$$\lambda_1 \leqq c\left(n - \frac{n-1}{2n-5}\right)$$

The equality holds if and only if $n = 4$ and M is congruent to the totally geodesic complex hypersurface Q^3 of the complex quadric $Q^4 = G_2(\mathbb{C}^4)$.

These two theorems are proved in the section 5. More detailed proofs of any our results are given in [8].

Notations. $M_{r,s}(\mathbb{C})$ denotes the set of all $r \times s$ matrices with entries in $\mathbb{C}$, and $M_r(\mathbb{C})$ stands for $M_{r,r}(\mathbb{C})$. I_r and O_r denote the identity r-matrix and the zero r-matrix.

§2. Preliminaries.

In this section, we discuss geometries of the complex r-plane Grassmann manifold and its first standard imbedding.

Let $M_r(\mathbb{C}^n)$ be the complex Stiefel manifold which is the set of all unitary r-systems of $\mathbb{C}^n$, i.e.,

$$M_r(\mathbb{C}^n) = \{Z \in M_{n,r}(\mathbb{C}) \mid Z^*Z = I_r\}.$$

The complex r-plane Grassman manifold $G_r(\mathbb{C}^n)$ is defined by

$$G_r(\mathbb{C}^n) = M_r(\mathbb{C}^n)/U(r).$$

The origin o of $G_r(\mathbb{C}^n)$ is defined by $\pi(Z_0)$, where $Z_0 = \begin{pmatrix} I_r \\ 0 \end{pmatrix}$ is a element of $M_r(\mathbb{C}^n)$, and $\pi\colon M_r(\mathbb{C}^n) \longrightarrow G_r(\mathbb{C}^n)$ is the natural projection.

The left action of the unitary group $\tilde{G} = SU(n)$ on $G_r(\mathbb{C}^n)$ is transitive, and the isotropy subgroup at the origin o is

$$\begin{aligned}\tilde{K} &= S(U(r)\cdot U(n-r)) \\ &= \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \,\middle|\, U_1 \in U(r),\ U_2 \in U(n-r),\ \det U_1 \det U_2 = 1 \right\}.\end{aligned}$$

so that $G_r(\mathbb{C}^n)$ is identified with a homogeneous space $\tilde{G}/\tilde{K}$

Set $\tilde{\mathfrak{g}} = \mathfrak{su}(n)$ and

$$\begin{aligned}\tilde{\mathfrak{k}} &= \mathbb{R} \oplus \mathfrak{su}(r) \oplus \mathfrak{su}(n-r) \\ &= \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} + a \begin{pmatrix} -\frac{1}{r}\sqrt{-1}I_r & 0 \\ 0 & \frac{1}{n-r}\sqrt{-1}I_{n-r} \end{pmatrix} \,\middle|\, a \in \mathbb{R},\ \begin{matrix} u_1 \in \mathfrak{su}(r) \\ u_2 \in \mathfrak{su}(n-r) \end{matrix} \right\},\end{aligned}$$

then $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{k}}$ are the Lie algebra of $\tilde{G}$ and $\tilde{K}$, respectively. Define a linear subspace $\tilde{\mathfrak{m}}$ of $\tilde{\mathfrak{g}}$ by

$$\tilde{\mathfrak{m}} = \left\{ \begin{pmatrix} 0 & -\xi^* \\ \xi & 0 \end{pmatrix} \,\middle|\, \xi \in M_{n-r,r}(\mathbb{C}) \right\},$$

then $\tilde{\mathfrak{m}}$ is identified with the tangent space $T_o(G_r(\mathbb{C}^n))$. The $\tilde{G}$-invariant complex structure J of $G_r(\mathbb{C}^n)$ and the $\tilde{G}$-invariant Kähler metric $\tilde{g}_c$ of $G_r(\mathbb{C}^n)$ of the maximal holomorphic sectional curvature c are given by

$$J\begin{pmatrix} 0 & -\xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-1}\xi^* \\ \sqrt{-1}\xi & 0 \end{pmatrix},$$

(2.1)
$$\tilde{g}_{c_o}(X, Y) = -\frac{2}{c} trXY, \quad X, Y \in \tilde{\mathfrak{m}}.$$

In the case of $r = 2$, the complex 2-plane Grassmann manifold $G_2(\mathbb{C}^n)$ admits another geometric structure named the quaternionic Kähler structure $\mathfrak{J}$. $\mathfrak{J}$ is a $\tilde{G}$-invariant subbundle of $End(T(G_2(\mathbb{C}^n)))$ of rank 3, where $End(T(G_2(\mathbb{C}^n)))$ is the $\tilde{G}$-invariant vector bundle of all linear endmorphisms of the tangent bundle $T(G_2(\mathbb{C}^n))$. Under the identification with $T_o(G_r(\mathbb{C}^n))$ and $\tilde{\mathfrak{m}}$, the fiber $\mathfrak{J}_o$ at the origin o is given by

$$\mathfrak{J}_o = \left\{ J_{\tilde{\varepsilon}} = ad(\tilde{\varepsilon}) \mid \tilde{\varepsilon} \in \tilde{\mathfrak{k}}_q \right\},$$

where $\tilde{\mathfrak{k}}_q$ is an ideal of $\tilde{\mathfrak{k}}$ defined by

$$\tilde{\mathfrak{k}}_q = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, u_1 \in \mathfrak{su}(2) \right\} \cong \mathfrak{su}(2).$$

Choose a basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of $\mathfrak{su}(2)$ satisfying $[\varepsilon_i, \varepsilon_{i+1}] = 2\varepsilon_{i+2}$, (mod 3) . Set $\tilde{\varepsilon}_i = \begin{pmatrix} \varepsilon_i & 0 \\ 0 & 0 \end{pmatrix}$ and $J_i = J_{\varepsilon_i}$ for $i = 1,2,3$, then the basis $\{J_1, J_2, J_3\}$ is a canonical basis of $\mathfrak{J}_o$, satisfying

$$J_i^2 = -id_{\tilde{\mathfrak{m}}} \quad \text{for } i = 1,2,3,$$
$$J_1J_2 = -J_2J_1 = J_3, \quad J_2J_3 = -J_3J_2 = J_1, \quad J_3J_1 = -J_1J_3 = J_2,$$
$$\tilde{g}_{c_o}(J_iX, J_iY) = \tilde{g}_{c_o}(X, Y), \quad \text{for } X, Y \in \tilde{\mathfrak{m}} \text{ and } i = 1,2,3.$$

There exists an element $\tilde{\varepsilon}_{\mathbb{C}}$ of the center of $\mathfrak{k}$ such that J is given by $J = ad(\tilde{\varepsilon}_{\mathbb{C}})$ on $\mathfrak{m}$. Therefore, J is commutable with $\mathfrak{J}$.

Let $HM(n,\mathbb{C})$ be the set of all Hermitian (n,n)-matrices over $\mathbb{C}$, which can be identified with $\mathbb{R}^{n^2}$. For $X, Y \in HM(n,\mathbb{C})$, the natural inner product is given by

$$(X,Y) = \frac{2}{c}\, trXY. \tag{2.2}$$

$GL(n,\mathbb{C})$ acts on $HM(n,\mathbb{C})$ by $X \longmapsto BXB^*$, $B \in GL(n,\mathbb{C})$, $X \in HM(n,\mathbb{C})$. Then the action of $SU(n)$ leaves the inner product (2.2) invariant.

The first standard imbedding Ψ of $G_r(\mathbb{C}^n)$ is defined by

$$\Psi(\pi(Z)) = ZZ^* \in HM(n,\mathbb{C}), \quad Z \in M_r(\mathbb{C}^n).$$

Ψ is $SU(n)$-equivariant and the image N of $G_r(\mathbb{C}^n)$ under Ψ is given as follows:

$$N = \Psi(G_r(\mathbb{C}^n)) = \{A \in HM(n,\mathbb{C}) \mid A^2 = A,\ trA = r\}. \tag{2.3}$$

The tangent bundle TN and the normal bundle $T^{\perp}N$ are given by

$$\begin{aligned} T_AN &= \{X \in HM(n,\mathbb{C}) \,|\, XA + AX = X\} \subset HM_0, \\ T_A^{\perp}N &= \{Z \in HM(n,\mathbb{C}) \,|\, ZA = ZX\}. \end{aligned} \tag{2.4}$$

In particular, at the origin $A_o = \Psi(o) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, we can obtain

$$\begin{aligned} T_{A_o}N &= \left\{ \begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} \,\middle|\, \xi \in M_{n-r,r}(\mathbb{C}) \right\}, \\ T_{A_o}^{\perp}N &= \left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \,\middle|\, Z_1 \in HM(r,\mathbb{C}),\ Z_2 \in HM(n-r,\mathbb{C}) \right\}. \end{aligned} \tag{2.5}$$

The complex structure J acts on $T_{A_o}N$ as follows:

$$J\begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{-1}\xi^* \\ \sqrt{-1}\xi & 0 \end{pmatrix}. \tag{2.6}$$

If $r = 2$, then the quaternionic Kähler structure $\mathfrak{J}$ acts on $T_{A_o}N$ as follows:

$$J_{\tilde{\varepsilon}}\begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon\xi^* \\ -\xi\varepsilon & 0 \end{pmatrix}, \quad \varepsilon \in \mathfrak{su}(2). \tag{2.7}$$

Let $\tilde{\sigma}$ and $\tilde{H}$ denote the second fundamental form and the mean curvature vector of Ψ, respectively. Then, for $A \in N$ and $X, Y \in T_AN$, we can see

$$\tilde{\sigma}_A(X,Y) = (XY + YX)(I - 2A) \tag{2.8}$$

$$\tilde{H}_A = \frac{c}{2r(n-r)}(rI - nA) \tag{2.9}$$

and $\tilde{\sigma}$ satisfies the following:

$$\tilde{\sigma}_A(JX, JY) = \tilde{\sigma}_A(X,Y), \tag{2.10}$$

$$(\tilde{\sigma}_A(X,Y),\, A) = -(X,\, Y). \tag{2.11}$$

§3. Examples.

One of the most simple typical examples of submanifolds of $G_r(\mathbb{C}^n)$ is a totally geodesic submanifold. B. Y. Chen and T. Nagano in [3, 4] determined maximal totally geodesic submanifolds of $G_2(\mathbb{C}^n)$. For arbitrary r, I. Satake and S. Ihara in [11, 5] determined all (equivariant) holomorphic imbeddings of a symmetric domain into another symmetric domain. Taking a compact dual symmetric space if necessary, we obtain the complete list of maximal totally geodesic Kähler submanifolds of $G_r(\mathbb{C}^n)$.

Since totally geodesic submanifols of $G_r(\mathbb{C}^n)$ are symmetric spaces, we can calculus the first eigenvalue of the Laplacian of M. (cf. [14])

Theorem 3.1. *Let M be a proper maximal totally geodesic Kähler submanifold of $G_r(\mathbb{C}^n)$, and λ_1 the first eigenvalue of the Laplace-Beltrami operator with respect to the induced Kähler metric. Then, M and λ_1 are one of the following (up to isomorphism).*

(1) $M_1 = G_r(\mathbb{C}^{n-1}) \hookrightarrow G_r(\mathbb{C}^n)$, $1 \leqq r \leqq n-2$, *and* $\lambda_1 = c(n-1)$

(2) $M_2 = G_{r-1}(\mathbb{C}^{n-1}) \hookrightarrow G_r(\mathbb{C}^n)$, $2 \leqq r \leqq n-1$, *and* $\lambda_1 = c(n-1)$

(3) $M_3 = G_{r_1}(\mathbb{C}^{n_1}) \times G_{r_2}(\mathbb{C}^{n_2}) \hookrightarrow G_{r_1+r_2}(\mathbb{C}^{n_1+n_2})$, $1 \leqq r_i \leqq n_i - 1$, $i = 1, 2$, *and* $\lambda_1 = c\min\{n_1, n_2\}$

(4) $M_4 = M_{4,p} = Sp(p)/U(p) \hookrightarrow G_p(\mathbb{C}^{2p})$, $p \geqq 2$, *and* $\lambda_1 = c(p+1)$

(5) $M_5 = M_{5,p} = SO(2p)/U(p) \hookrightarrow G_p(\mathbb{C}^{2p})$, $p \geqq 4$, *and* $\lambda_1 = c(p-1)$

(6) $M_{6,m} = \mathbb{C}P^p \hookrightarrow G_r(\mathbb{C}^n)$: *the complex projective space,*
$r = \binom{p}{m-1}$, $n = \binom{p+1}{m}$, $2 \leqq m \leqq p-1$,
and $\lambda_1 = c(p+1)\binom{p-1}{m-1}^{-1}$

(7) $M_7 = Q^3 \hookrightarrow Q^4 = G_2(\mathbb{C}^4)$: *the complex quadric, and* $\lambda_1 = 3c$

(8) $M_8 = M_{8,2l} = Q^{2l} \hookrightarrow G_r(\mathbb{C}^{2r})$: *the complex quadric,* $r = 2^{l-1}$, $l \geqq 3$, *and* $\lambda_1 = c\dfrac{2l}{2^{l-2}}$

In above list, notice that $M_{4,2} = M_7$ and $M_{5,4} = M_{8,6}$.

Another one of the most simple typical examples of submanifolds of $G_r(\mathbb{C}^n)$ is a homogeneous Kähler hypersurface. K. Konno in [6] determined all Kähler C-spaces embedded as a hypersurface into a Kähler C-space with the second Betti number $b_2 = 1$.

Theorem 3.2. *Let M be a compact, simply connected homogeneous Kähler hypersurface of $G_r(\mathbb{C}^n)$, and λ_1 the first eigenvalue of the Laplace-Beltrami operator with respect to the induced Kähler metric. Then, M and λ_1 are one of the following (up to isomorphism).*

(1) $M_9 = \mathbb{C}P^{n-2} \hookrightarrow \mathbb{C}P^{n-1} = G_1(\mathbb{C}^n)$ *and* $\lambda_1 = c(n-1)$
(2) $M_{10} = Q^{n-2} \hookrightarrow \mathbb{C}P^{n-1} = G_1(\mathbb{C}^n)$ *and* $\lambda_1 = c(n-2)$
(3) $M_7 = Q^3 \hookrightarrow Q^4 = G_2(\mathbb{C}^4)$ *and* $\lambda_1 = 3c$
(4) $M_{11} = Sp(l)/U(2)Sp(l-2) \hookrightarrow G_l(\mathbb{C}^{2l})$: *Kähler C-space of type* (C_l, α_2), $l \geqq 2$ *and* $\lambda_1 = c(2l-1)$

M_9 and M_7 are totally geodesic. M_9 M_{10} and M_7 are symmetric spaces. If $l = 2$, then M_{11} is congruent to M_7.

For each l with $l > 2$, M_{11} is not a symmetric space. Then, it is not easy to calculus the first eigenvalue λ_1 of M_{11}. We will calculus λ_1 of M_{11} in the next section.

From these two theorems, we obtain the following proposition:

Proposition 3.3. *Let M be either a proper maximal totally geodesic Kähler submanifold of $G_r(\mathbb{C}^n)$ or a compact simply connected homogeneous Kähler hypersurface of $G_r(\mathbb{C}^n)$. Then, the first eigenvalue λ_1 of M with respect to the induced Kähler metric satisfies the following inequality:*

$$\lambda_1 \leqq c\,(n-1).$$

Moreover, the equality holds if and only if M is congruent to one of the follows:

$$M_1, \quad M_2, \quad M_{4,2} = M_7, \quad M_9, \quad M_{11}.$$

§4. the homogeneous Kähler hypersurface (C_l, α_2).

In this section, we will consider the first eigenvalue of the Kähler C-space of type (C_l, α_r). For details, see [2] and [13].

The Kähler C-space of type (C_l, α_r) is a compact simply connected homogeneous Kähler manifold $M = G/K = Sp(l)/U(r) \cdot Sp(l-r)$, $1 \leqq r \leqq l$. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ Lie algebras of G and K, respectively, i.e.,

$$\mathfrak{g} = \mathfrak{sp}(l) = \left\{ \begin{pmatrix} A & -\overline{C} \\ C & \overline{A} \end{pmatrix} \,\middle|\, \begin{array}{c} A, C \in M_l(\mathbb{C}), \\ A^* = -A,\ {}^tC = C \end{array} \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A' & 0 & -\overline{C'} \\ 0 & 0 & \overline{A} & 0 \\ 0 & C' & 0 & \overline{A'} \end{pmatrix} \;\middle|\; \begin{array}{c} A \in M_r(\mathbb{C}), \\ A', C' \in M_{l-r}(\mathbb{C}), \\ A^* = -A,\ A'^* = -A',\ {}^tC' = C' \end{array} \right\}$$
$$= \mathfrak{u}(r) + \mathfrak{sp}(l-r).$$

$\mathfrak{g}$ is a compact semisimple Lie algebra of type C_l.

For $x, y \in M_{l-r,r}(\mathbb{C})$ and $z \in M_r(\mathbb{C})$ with ${}^tz = z$, define

$$\eta(x,y,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ z & {}^ty & 0 & -{}^tx \\ y & 0 & 0 & 0 \end{pmatrix}.$$

Note that, if $r = l$, then we ignore x and y, and $\eta(x,y,z)$ and $\eta(0,0,z)$ denote a matrix $\begin{pmatrix} 0_l & 0_l \\ z & 0_l \end{pmatrix}$, $z \in M_l(\mathbb{C})$, ${}^tz = z$.

Let $\mathfrak{m}$, $\mathfrak{m}^+$ and $\mathfrak{m}^-$ be subspaces of $\mathfrak{g}$ defined by

$$\begin{aligned} \mathfrak{m} &= \{\eta(x,y,z) - \eta(x,y,z)^*\}, \\ \mathfrak{m}^+ &= \{\eta(x,y,z)\}, \\ \mathfrak{m}^- &= \{\eta(x,y,z)^*\}, \end{aligned}$$

so that $\mathfrak{m}$, $\mathfrak{m}^+$ and $\mathfrak{m}^-$ are K-invariant under the adjoint action, and $\mathfrak{m}$ is identified with the tangent space T_oM of M at the origin $o = \{K\}$. Moreover, the complexification $\mathfrak{m}^{\mathbb{C}}$ of $\mathfrak{m}$ is the direct sum $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ + \mathfrak{m}^-$, and $\mathfrak{m}^{\pm}$ is the $\pm\sqrt{-1}$-eigenspace of the complex structure J of M at the origin o.

For any positive real number a, the Einstein-Kähler metric $g(a)$ of M is given by

$$(4.1) \qquad g(a)(X,X) = 2a\, tr(x^*x + y^*y + \bar{z}z), \quad X = \eta(x,y,z) - \eta(x,y,z)^* \in \mathfrak{m}.$$

Relative to this metric, the scalar curvature τ of M is given by

$$\tau = \frac{2(2l - r + 1)}{a} \dim_{\mathbb{C}} M.$$

Y. Matsushima and M. Obata showed the following:

Theorem 4.1 [9]. *Let M be an n-dimensional compact Einstein Kähler manifold of positive scalar curvature τ. Then the first eigenvalue $\lambda_1(M)$ of the Laplacian satisfies that*

$$\lambda_1(M) \geqq \frac{\tau}{n}.$$

The equality holds if and only if M admits an one-parameter group of isometries (i.e., a non-trivial Killing vector field).

The natural inclusion $Sp(l) \mapsto SU(2l)$ defines an immersion φ of M into $\tilde{M} = G_r(\mathbb{C}^{2l}) = \tilde{G}/\tilde{K} = SU(2l)/S(U(r) \cdot U(2l-r))$ by

$$\varphi(g \cdot K) = g \cdot \tilde{K}, \quad g \in G.$$

Under identification of $T_o\tilde{M}$ with $\tilde{\mathfrak{m}}$, the image of $X = \eta(x, y, z) - \eta(x, y, z)^* \in \mathfrak{m}$ is

$$\varphi_*(X) = \begin{pmatrix} 0 & -x^* & -\bar{z} & -y^* \\ x & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ y & 0 & 0 & 0 \end{pmatrix},$$

so that we have

$$\tilde{g}_c(\varphi_*(X), \varphi_*(X)) = \frac{4}{c}\, tr(x^*x + y^*y + \bar{z}z). \tag{4.2}$$

Therefore, Theorem 4.1, (4.1) and (4.2) imply the following.

Theorem 4.2. *For the Kähler C-space $M = Sp(l)/U(r) \cdot Sp(l-r)$ of type (C_l, α_r) equiped with the Kälher metric $g(\frac{2}{c})$, M is immersed to $G_r(\mathbb{C}^{2l})$ by the Kähler immersion φ. The complex dimension, and the first eigenvalue $\lambda_1(M)$ of the Laplacian are given by*

$$\dim_{\mathbb{C}} M = \frac{r(4l - 3r + 1)}{2}, \quad \lambda_1(M) = c\,(2l - r + 1).$$

In particular, if $r = 2$, then $M = Sp(l)/U(2) \cdot Sp(l-2)$ is a Kähler hypersurface of $G_2(\mathbb{C}^{2l})$, whose first eigenvalue $\lambda_1(M)$ of the Laplacian is given by

$$\lambda_1(M) = c\,(2l - 1).$$

For $z \in M_r(\mathbb{C})$, define an unit vector ν at the origin o of $G_2(\mathbb{C}^{2l})$ by

$$\nu(z) = \begin{pmatrix} 0 & 0 & -z^* & 0 \\ 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \tilde{\mathfrak{m}}, \quad \frac{4}{c}\, tr\, z^*z = 1.$$

Then $\nu(z)$ is tangent to M if and only if z is symmetric.

The Kähler hypersurface $M = (C_l, \alpha_2)$ satisfies the following property relative to the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{2l})$.

Proposition 4.3. *The Kähler hypersurface $M = Sp(l)/U(2) \cdot Sp(l-2)$ of $G_2(\mathbb{C}^{2l})$ satisfies*

$$\mathfrak{J}\, T^{\perp}M \subset TM \quad (\Longleftrightarrow J\xi \perp \mathfrak{J}\xi \text{ for any } \xi \in T^{\perp}M\,), \tag{4.3}$$

where TM and $T^{\perp}M$ are the tangent bundle and the normal bundle of M, respectively.

Proof. Let ν_o be an unit normal vector of M at o defined by

$$\nu_o = \nu(z_o), \quad z_o = \frac{1}{2}\sqrt{\frac{c}{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that the normal space $T_o^\perp M$ is given by

$$T_o^\perp M = \mathbb{R}\left\{\nu_o,\ J\nu_o = \nu(\sqrt{-1}z_o)\right\}.$$

Then we see

$$\begin{aligned}\mathfrak{J}_o\, T_o^\perp M =& \mathbb{R}\left\{J_i\nu_o,\ J_iJ\nu_o, \quad i=1,2,3\right\}\\ =& \mathbb{R}\left\{\nu(z_o\varepsilon_i),\ \nu(\sqrt{-1}z_o\varepsilon_i), \quad i=1,2,3\right\},\end{aligned}$$

where J_1, J_2 and J_3 are a canonical basis of $\mathfrak{J}_o$ defined in the section 2. It is easy to check that $z_o\varepsilon_i$ and $\sqrt{-1}z_o\varepsilon_i$ are symmetric, so that we obtain

$$\mathfrak{J}_o\, T_o^\perp M \subset T_o M.$$

Since the quaternionic Kähler structure $\mathfrak{J}$ is $\tilde{G}$-invariant, and since the immersion φ is G-equivariant, (4.3) holds at any point of M. □

If the ambient space is $G_2(\mathbb{C}^4)$, then the condition (4.3) determines a Kähler hypersurface as follows:

Proposition 4.4. *Suppose that a Kähler hypersurface M of $Q^4 = G_2(\mathbb{C}^4)$ satisfies the condition*

$$\mathfrak{J}\, T^\perp M \subset TM.$$

Then M is totally geodesic. Moreover, if M is compact, then M is congruent to a complex quadric $Q^3 = Sp(2)/U(2)$.

Proof. Denote by $\tilde{\nabla}$ the Riemannian connection of Q^4, and denote by ∇, σ, A and $\nabla^\perp$, the Riemannian connection, the second fundamental form, the shape operator, and the normal connection of M, respectively. It is well-known that Gauss' formula and Weingarten's formula hold:

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + \sigma(X,Y),\\ \tilde{\nabla}_X \xi &= -A_\xi X + \nabla^\perp_X \xi,\end{aligned} \tag{4.4}$$

for $X, Y \in TM$ and $\xi \in T^\perp M$. The metric condition implies

$$\tilde{g}_c(\sigma(X,Y),\xi) = \tilde{g}_c(A_\xi X, Y). \tag{4.5}$$

Relative to the complex structure J, σ and A satisfy

$$\sigma(X,JY) = J\sigma(X,Y), \quad A_\xi \circ J = -J\circ A_\xi = -A_{J\xi}. \tag{4.6}$$

For a local unit normal vector field ξ, we define local vector fields as follow: $e_i = J_i\xi$, $i = 1,2,3$, where J_1, J_2 and J_3 are a local canonical basis of $\mathfrak{J}$. Then, under the assumption of this proposition, $\{e_1, e_2, e_3, Je_1, Je_2, Je_3, \xi, J\xi\}$ is a local orthonormal frame field of Q^4 such that $\{e_1, e_2, e_3, Je_1, Je_2, Je_3\}$ is a tangent frame of M. For $X \in TM$, (4.4) implies

$$\begin{aligned}(4.7)\qquad \nabla_X e_i + \sigma(X, e_i) = \tilde{\nabla}_X e_i &= (\tilde{\nabla}_X J_i)\xi + J_i(\tilde{\nabla}_X \xi)\\ &= (\tilde{\nabla}_X J_i)\xi - J_i A_\xi X + J_i(\nabla^{\perp}_X \xi)\end{aligned}$$

Since $\mathfrak{J}$ is parallel with respect to the connection $\tilde{\nabla}$, we have $\tilde{\nabla}_X J_i \in \mathfrak{J}$, so that the normal component of (4.7) is

$$\begin{aligned}\sigma(X, e_i) &= -\tilde{g}_c(J_i A_\xi X, \xi)\xi - \tilde{g}_c(J_i A_\xi X, J\xi)J\xi\\ &= g_c(A_\xi X, e_i)\xi + g_c(A_\xi X, Je_i)J\xi,\end{aligned}$$

where g_c is the induced Kähler metric of M. On the other hand, (4.5) and (4.6) imply

$$\begin{aligned}\sigma(X, e_i) &= \tilde{g}_c(\sigma(X, e_i), \xi)\xi + \tilde{g}_c(\sigma(X, e_i), J\xi)J\xi\\ &= g_c(A_\xi X, e_i)\xi - g_c(A_\xi X, Je_i)J\xi.\end{aligned}$$

From these two equations, we get

$$(4.8)\qquad g_c(A_\xi X, Je_i) = 0.$$

Instead of X, applying to JX, we have

$$g_c(A_\xi X, e_i) = g_c(-A_\xi JX, Je_i) = 0.$$

Therefore, we have $A_\xi = 0$, or $\sigma = 0$, so that M is totally geodesic. By B. Y. Chen and T. Nagano [3]'s results, if M is compact, M is congruent to a complex quadric $Q^3 = Sp(2)/U(2)$. □

§5. proof of main theorems.

Let M be a compact connected Kähler hypersurface of $G_r(\mathbb{C}^n)$ immersed by a immersion φ. It is well-known that every $HM(n, \mathbb{C})$-valued function F satisfies

$$(5.1)\qquad (\Delta F, \Delta F)_{L^2} - \lambda_1(\Delta F, F)_{L^2} \geqq 0$$

The equality holds if and only if F is a sum of eigenfunctions with respect to eigenvalues 0 and λ_1. It is equivalent to that there exists a constant vector $C \in HM(n, \mathbb{C})$ such that $\Delta(F - C) = \lambda_1(F - C)$.

Denote by H the mean curvature vector of the isometric immersion $\Phi = \Psi \circ \varphi$. Then, since M is minimal in $G_r(\mathbb{C}^n)$, (2.9) implies

$$\begin{aligned}(5.2)\qquad 2(r(n-r)-1)H_A &= 2r(n-r)\tilde{H}_A - \tilde{\sigma}_A(\xi, \xi) - \tilde{\sigma}_A(J\xi, J\xi)\\ &= c(rI - nA) - \tilde{\sigma}_A(\xi, \xi) - \tilde{\sigma}_A(J\xi, J\xi),\end{aligned}$$

where A is a position vector of $\Phi(M)$ in $HM(n, \mathbb{C})$, and ξ is a local unit normal vector field of φ. Using (2.11) and (5.2) , we get

$$(5.3)\qquad (H_A, A) = -1.$$

$HM(n, \mathbb{C})$-valued function Φ satisfies $\Delta\Phi = -2(r(n-r)-1)H$, so that (5.1) and (5.3) imply the following. The equality condition dues to T. Takahashi's theorem in [12].

Lemma 5.1.

$$2(r(n-r)-1)\int_M (H_A, H_A)\, dv_M - \lambda_1 vol(M) \geqq 0. \tag{5.4}$$

The equality holds if and only if Φ is a minimal immersion of M into some round sphere in $HM(n,\mathbb{C})$, more precisely, there exists some positive constat R and some constant vector $C \in HM(n,\mathbb{C})$ such that H_A satisfies

$$H_A = \frac{1}{R^2}(C-A). \tag{5.5}$$

Lemma 5.2. *If the equality holds in (5.4) , then M is contained in a totally geodesic submanifold of $G_r(\mathbb{C}^n)$ which is product of Grassmann manifolds, more precisely, there exist integers k_i, r_i, $i = 1, \cdots, m$ such that*

$$0 \leqq r_i \leqq k_i, \qquad r_1 \geqq r_2 \geqq \cdots \geqq r_m,$$

$$\sum_{i=1}^m r_i = r, \qquad \sum_{i=1}^m k_i = n,$$

$$M \subset G_{r_1}(\mathbb{C}^{k_1}) \times G_{r_2}(\mathbb{C}^{k_2}) \times \cdots \times G_{r_m}(\mathbb{C}^{k_m}) \subset G_r(\mathbb{C}^n). \tag{5.6}$$

Notice that $G_0(\mathbb{C}^{k_i}) = G_{k_i}(\mathbb{C}^{k_i}) = \{\text{one point}\}$.

proof. Assume that this equality holds in (5.4) .

Since M is minimal in $G_r(\mathbb{C}^n)$, H is normal to $G_r(\mathbb{C}^n)$. Then, from (2.4) and (5.5) , we get

$$CA = AC, \tag{5.7}$$

where C is a constant vector in Lemma 5.1. Since $SU(n)$ acts on $G_r(\mathbb{C}^n)$ transitively, without loss of generalization, we can assume that C is a diagonal matrix as follows:

$$C = \begin{pmatrix} c_1 I_{k_1} & & & 0 \\ & c_2 I_{k_2} & & \\ & & \ddots & \\ 0 & & & c_m I_{k_m} \end{pmatrix}, \quad k_i > 0, \quad c_i \neq c_j \,(i \neq j). \tag{5.8}$$

Notice that

$$n = k_1 + k_2 + \cdots + k_m.$$

Define a linear subspace L of $HM(n,\mathbb{C})$ by $L = \{Z \in HM(n,\mathbb{C}) \mid ZC = CZ\}$, so that

$$L = \left\{ \begin{pmatrix} Z_1 & & & 0 \\ & Z_2 & & \\ & & \ddots & \\ 0 & & & Z_m \end{pmatrix} \,\middle|\, Z_i \in M_{k_i}(\mathbb{C}) \right\}.$$

From (5.7) , M is contained in $G_r(\mathbb{C}^n) \cap L$.

For each integer r_i with $0 \leqq r_i \leqq k_i$, $\sum_{i=1}^m r_i = r$, let's define connected subsets of $G_r(\mathbb{C}^n)$ by

$$W_{r_1,\cdots,r_m} = \left\{ \begin{pmatrix} A_1 & & 0 \\ & A_2 & \\ & & \ddots \\ 0 & & & A_m \end{pmatrix} \;\middle|\; \begin{matrix} A_i \in M_{k_i}(\mathbb{C}), \\ A_i^2 = A_i, \quad tr\, A_i = r_i \end{matrix} \right\}.$$

So, $G_r(\mathbb{C}^n) \cap L$ is a disjoint union of all $W_{r_1,\cdots,r_m}$'s. Since M is connected, M is contained in suitable one of $W_{r_1,\cdots,r_m}$'s, saying $W_{r_1,\cdots,r_m}$. By the definition, we see

$$W_{r_1,\cdots,r_m} = G_{r_1}(\mathbb{C}^{k_1}) \times G_{r_2}(\mathbb{C}^{k_2}) \times \cdots \times G_{r_m}(\mathbb{C}^{k_m}).$$

Without loss of generalization, we can choose a diagonal matrix C with respect to which the inequalities $r_1 \geqq r_2 \geqq \cdots \geqq r_m$ hold. □

From (2.8) , (2.10) and (5.2) , we get

$$\text{(5.9)} \qquad H_A = \frac{c}{2(r(n-r)-1)} \left\{ (rI - nA) - \frac{4}{c}(\Psi_*\xi)^2(I-2A) \right\}.$$

Using (2.2) and (2.3) , we see

(5.10)

$$(H_A, H_A) = \frac{c}{2(r(n-r)-1)^2} \left\{ nr(n-r) - 2\, tr\, \frac{4}{c}\, r\, (\Psi_*\xi)^2 \left(I + \frac{n-2r}{r} A \right) \right.$$

$$\left. + \, tr\, \frac{16}{c^2} (\Psi_*\xi)^2 (I-2A)(\Psi_*\xi)^2(I-2A) \right\}.$$

Since the immersion Ψ is $\tilde{G}$-equivariant, for any $A \in \Phi(M)$, there exists a element $g_A \in \tilde{G}$ and a matrix $v_A \in M_{n-r,r}(\mathbb{C})$ satisfying $A_o = g_A A g_A^*$ and

$$\text{(5.11)} \qquad \sqrt{\frac{c}{4}} \begin{pmatrix} 0 & v_A^* \\ v_A & 0 \end{pmatrix} = g_A(\Psi_*\xi)\, g_A^* \,.$$

Since the inner product (,) is $\tilde{G}$-equivariant and ξ is unit, we have $tr\, v_A^* v_A = tr\, v_A v_A^* = 1$. After translating by g_A, together with (5.11) , (5.10) implies

$$\text{(5.12)} \qquad (H_A, H_A) = \frac{c}{2(r(n-r)-1)^2} \{ n(r(n-r)-2) + 2\, tr\; (v_A^* v_A v_A^* v_A) \}.$$

Lemma 5.3. *(a) For $v \in M_{n-r,r}(\mathbb{C})$ with $tr\, v^*v = 1$, the following inequality holds*

$$tr\, v^*vv^*v \leqq 1. \tag{5.13}$$

(b) Moreover, next three conditions are equivalent to each other.

(1) *The equality holds in* (5.13)

(2) *The hermitian r-matrix v^*v is similar to* $\begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix}$.

(3) *The hermitian $(n-r)$-matrix vv^* is similar to* $\begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}$.

(c) If the equality holds in (5.13) *, then there exists $R = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \in S(U(r) \cdot U(n-r))$ such that $v' = QvP^*$ satisfies*

$$v'^*v' = \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \quad \text{and} \quad v'v'^* = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}.$$

Proof. Lemma 5.3 follows from that both of hermitian matrices v^*v and vv^* are similar to diagonal matrices with non-negative eigenvalues.

Form (5.12) and Lemma 5.3, the following lemma is immediately obtained, which is used to prove Theorem A.

Lemma 5.4.

$$(H_A, H_A) \leqq \frac{c}{2(r(n-r)-1)} \left\{ n - \frac{n-2}{r(n-r)-1} \right\}. \tag{5.14}$$

The equality holds if and only if, for any $A \in \Phi(M)$, it is possible to choose v_A satisfying

$$v_A^* v_A = \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \quad \text{and} \quad v_A v_A^* = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}. \tag{5.15}$$

proof of Theorem A. (5.4) and (5.14) imply

$$\lambda_1 \leqq c \left(n - \frac{n-2}{r(n-r)-1} \right).$$

Let's assume that this equality holds. Then, the equality conditions of Lemmas 5.1 and 5.4 hold.

Assume $m = 1$. Then, (5.5) and (5.9) imply

$$\frac{1}{R^2}(c_1 I - A) = \frac{c}{2(r(n-r)-1)} \left\{ (rI - nA) - \frac{4}{c}(\Psi_* \xi)^2 (I - 2A) \right\}.$$

After translating by g_A, together with (5.11) and (5.15) , we obtain

$$\begin{aligned}\frac{1}{R^2}(c_1-1)I_r =& \frac{c}{2(r(n-r)-1)}\left\{(r-n)I_r + \begin{pmatrix}1 & 0\\ 0 & 0_{r-1}\end{pmatrix}\right\},\\ \frac{1}{R^2}c_1 I_{n-r} =& \frac{c}{2(r(n-r)-1)}\left\{rI_{n-r} - \begin{pmatrix}1 & 0\\ 0 & 0_{n-r-1}\end{pmatrix}\right\}.\end{aligned}$$

The first equation implies $r=1$, and the second one implies $n-r=1$. So, we have $n=2$ and $r=1$. This contradicts that M is a complex hypersurface.

Since $m \geqq 2$, from Lemma 5.2, M is contained in a proper totally geodesic submanifold of $G_r(\mathbb{C}^n)$. On the other hand, M is of complex codimension 1 in $G_r(\mathbb{C}^n)$. Consequently, either $r=1$ or $r=n-1$ occurs, and M is a totally geodesic complex hypersurface of a complex projective space $\mathbb{C}P^{n-1} \cong G_1(\mathbb{C}^n) \cong G_{n-1}(\mathbb{C}^n)$. $\square$

Proof of Theorem B. Let's assume that M is a compact connected Kähler hypersurface of $G_2(\mathbb{C}^n)$ satisfying the condition $J\xi \perp \mathfrak{J}\xi$. Since both of the complex structure and the quaternionic Kähler structure are $\tilde{G}$-invariant, we obtain, at the origin A_o,

$$J\begin{pmatrix}0 & v_A^*\\ v_A & 0\end{pmatrix} \perp J_i\begin{pmatrix}0 & v_A^*\\ v_A & 0\end{pmatrix}, \quad i=1,2,3, \tag{5.16}$$

where J_1, J_2 and J_3 are a canonical basis of $\mathfrak{J}_o$ defined in the section 2. Set

$$v_A = (\,v_A' \quad v_A''\,), \quad v_A', v_A'' \in M_{n-2,1}(\mathbb{C}) \cong \mathbb{C}^{n-2}.$$

Using (2.6) and (2.7) , (5.16) implies that $|v_A'| = |v_A''|$ and $v_A' \perp v_A''$. Combing them with $tr\, v_A^* v_A = 1$, we obtain $|v_A'| = |v_A''| = \frac{1}{\sqrt{2}}$, so that

$$v_A^* v_A = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{5.17}$$

Together with (5.17) , (5.12) implies

$$(H_A, H_A) = \frac{c}{2(2n-5)}\left\{n - \frac{n-1}{2n-5}\right\}.$$

Therefore, form Lemma 5.1, we obtain

$$\lambda_1 \leqq c\left(n - \frac{n-1}{2n-5}\right).$$

Let's assume that this equality holds. Then, the equality conditions of Lemma 5.1. holds.

Computing dimensions of manifolds in (5.6), we have

$$2n - 5 \leqq \sum_{i=1}^{m} r_i(k_i - r_i)\,. \tag{5.18}$$

From $\sum_{i=1}^{m} r_i = 2$ and $r_1 \geqq r_2 \geqq \cdots \geqq r_m$, the following two cases occur:

$$\begin{aligned} &\text{Case I}: \quad r_1 = r_2 = 1, \quad r_3 = \cdots = r_m = 0, \\ &\text{Case II}: \quad r_1 = 2, \quad r_2 = \cdots = r_m = 0. \end{aligned}$$

In Case I, (5.18) implies $2n - 5 \leqq k_1 + k_2 - 2 \leqq n - 2$, so $n \leqq 3$. This is contradiction.

Therefore, Case II occurs. Then, (5.18) implies $2n - 5 \leqq 2(k_1 - 2)$, so that we have $n = k_1$, $m = 1$, $k_2 = \cdots = k_m = 0$. (5.5) and (5.9) imply

$$\frac{1}{R^2}(c_1 I - A) = \frac{c}{2(2n-5)}\left\{(2I - nA) - \frac{4}{c}(\Psi_*\xi)^2(I - 2A)\right\}.$$

After translating by g_A, together with (5.11) and (5.17), we obtain

$$\begin{aligned} \frac{1}{R^2}(c_1 - 1) &= \frac{c}{2(2n-5)}\left\{2 - n + \frac{1}{2}\right\}, \\ \frac{1}{R^2} c_1 I_{n-2} &= \frac{c}{2(2n-5)}\{2I_{n-2} - v_A v_A^*\}. \end{aligned}$$

The second equation implies

$$v_A v_A^* = d I_{n-2}, \quad d = 2 - \frac{2(2n-5)}{c}\frac{c_1}{R^2}\,. \tag{5.19}$$

From (5.17), we have

$$d\, v_A = d I_{n-2} v_A = (v_A v_A^*) v_A = v_A (v_A^* v_A) = \frac{1}{2} v_A,$$

so that $d = \frac{1}{2}$. Consequently, taking traces of both sides of (5.19), we obtain $n = 4$.

Therefore, from Proposition 4.4, M is congruent to Q^3. $\square$

References

[1] J-P. Bourguignon, P. Li and S. T. Yau, *Upper bound for the first eigenvalue of algebraic submanifolds*, Comment. Math. Helv. **69** (1994), 199-207.

[2] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces I*, Amer. J. Math. **80** (1958), 458-538.

[3] B. Y. Chen and T. Nagano, *Totally geodesic submanifolds of symmetric spaces, I*, Duke Math. J. **44** (1977), 745-755.

[4] B. Y. Chen and T. Nagano, *Totally geodesic submanifolds of symmetric spaces, II*, Duke Math. J. **45** (1978), 405-425.

[5] S. Ihara, *Holomorphic imbeddings of symmetric domains*, J. Math. Soc. Japan **19** (1967), 261-302.

[6] K. Konno, *Homogeneous hypersurfaces in Kähler C-spaces with* $b_2 = 1$, J. Math. Soc. Japan **40** (1988), 687-703.

[7] Y. Miyata, *On an upper bound for the first eigenvalue of Kähler submanifolds of* $\mathbb{C}P^n$, Bulletin of the Liberal Arts & Science Course **21** (1993), Nihon University School of Medicine, 13-23.

[8] Y. Miyata, *Spectral geometry of Kähler hypersurfaces in a complex Grassmann manifold* (to appear).

[9] M. Obata, *Riemannian manifolds admitting a solution of a certain system of differential equations*, Proc. U.S.-Japan Sem. in Differential Geom., Kyoto, Japan (1965), 101-114.

[10] A. Ros, *On spectral geometry of Kaehler submanifolds*, J. Math. Soc. Japan **36** (1984), 433-448.

[11] I. Satake, *Holomorphic imbeddings of symmetric domains into a Siegel space*, Amer. J. Math. **87** (1965), 425-461.

[12] T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380-385.

[13] M. Takeuchi, *Homogeneous Kähler submanifolds in complex projective spaces*, Japan.J. Math. **4** (1978), 171-219.

[14] M. Takeuchi, *Modern Spherical Functions*, Translations of Mathematical Monographs, vol. 135, Amer. Math. Soc., 1994.